*J. Fluid Mech.* (1970). *vol.* 43, *part* 4, *pp.* 641–660
*Printed in Great Britain*



# Pressure drop due to the motion of neutrally buoyant particles in duct flows

By HOWARD BRENNER
Department of Chemical Engineering, Carnegie–Mellon University, Pittsburgh, Pennsylvania

(Received 15 December 1969 and in revised form 4 May 1970)

The additional pressure drop arising from the presence of a neutrally-buoyant, eccentrically positioned, spherical particle in a Poiseuille flow is calculated to $O(a/R_o)^5$ ($a$ = sphere radius; $R_o$ = tube radius). Similar calculations (of a lesser order of accuracy) are given for non-circular conduits and for ellipsoidal particles. Due to changes in particle orientation resulting from rotation, the instantaneous pressure drop for an ellipsoid of revolution varies periodically with time. This pressure diminution is averaged over one period to obtain the time-average pressure drop.

## 1. Introduction

Knowledge of the pressure drop accompanying the motion of neutrally buoyant particles suspended in a tube flow is relevant to the flow of suspensions, blood flow, and capillary tube rheology. Hochmuth & Sutera (1970) treat the case of a single spherical particle moving along the axis of a circular tube for both small and large ratios of particle-to-tube radii. Large, closely fitted, spherical and spheroidal particles in circular tubes have been analysed by Bungay & brenner (1970), using matched asymptotic expansion methods. Concentric, as well as eccentric particle positions were studied. Wang & Skalak (1969) examined the case of an infinite line of spheres translating along the tube axis for an extensive range of particle-to-tube radius ratios and centre-to-centre axial cing ratios. Their work was subsequently extended by Chen & Skalak (1970) to spheroidal particles, again for the concentric case.

A calculation is presented in this paper of the additional pressure drop caused by the movement of an eccentrically situated, neutrally buoyant particle in a duct, for small values of the ratio of particle-to-duct size. Results are given explicitly for spherical, and (general) ellipsoidal particles in ducts of arbitrary cross-sectional shape. In contrast with the comparable axisymmetric calculations of Hochmuth & Sutera (1970) and Skalak and co-workers (1969, 1970), the calculation is brought to completion without the need for a detailed solution satisfying the requisite boundary conditions on the duct wall.

## 2. Formulation of the problem

In this section a general technique is developed for computing the additional pressure drop arising from the presence of a particle in an otherwise unidirectional duct flow. The method is illustrated by the explicit example of a spherical particle in a circular tube.

Consider a spherical particle (radius $= a$) located at an arbitrary radial position within a circular duct (radius $= R_o$) within which a Poiseuille flow is occurring (mean velocity $= V_m$). The geometrical configuration is shown in figure 1.

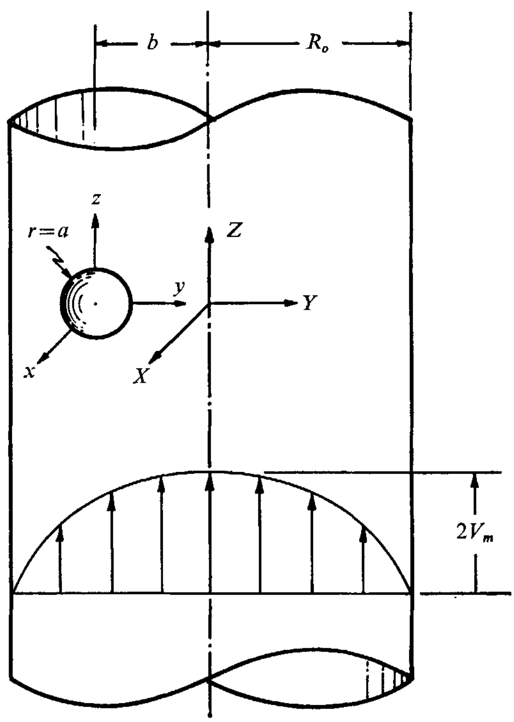


FIGURE 1. Eccentrically positioned sphere in a circular cylinder.

$(X, Y, Z)$ are a right-handed system of Cartesian co-ordinates with origin along the cylinder axis and the $Z$ axis pointing in the direction of net flow. The negative $Y$ axis passes through the sphere centre, whose instantaneous co-ordinates are $(X = 0,\ Y = -b,\ Z = 0)$. The radial distance $R$ from the tube axis to any point is $R^2 = X^2 + Y^2$. Let $(x, y, z)$ be a second Cartesian system, having its origin at the sphere centre, and chosen such that $x = X$, $y = Y+b$, and $z = Z$. Finally, introduce the system of spherical polar co-ordinates $(r, \theta, \phi)$

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \tag{2.1}$$

having its origin at the sphere centre.

It will eventually be assumed that the sphere is not too near to the wall, i.e. that

$$a/(R_o - b) \ll 1. \tag{2.2}$$

In the absence of the sphere, the undisturbed Poiseuille flow $(\mathbf{v}^o, p^o)$ is given by

$$\mathbf{v}^o = \hat{\mathbf{z}} v^o, \tag{2.3}$$

$$v^o = 2V_m[1-(R/R_o)^2], \tag{2.4}$$

$$p^o = \text{const.} - 8\mu_o V_m z/R_o^2, \tag{2.5}$$

where the caret denotes a unit vector; $\mu_o$ is the fluid viscosity. The arbitrary constant in the pressure field may be set equal to zero without loss of generality. This flow satisfies the equations

$$\nabla^2 \mathbf{v}^o = \mu_o^{-1} \nabla p^o, \quad \nabla \cdot \mathbf{v}^o = 0 \tag{2.6}$$

and is simultaneously a Stokes (creeping) flow and a Navier–Stokes flow since the inertial terms vanish identically.

When the sphere is present in the duct the fluid motion is altered. Let $(\mathbf{v}, p)$ denote the flow in the presence of the sphere (at the same mean velocity $V_m$). It is assumed that this motion is governed by Stokes' equations,

$$\nabla^2 \mathbf{v} = \mu_o^{-1} \nabla p, \quad \nabla \cdot \mathbf{v} = 0. \tag{2.7}$$

The instantaneous boundary conditions are

$$\mathbf{v} = \mathbf{U}_o + \mathbf{\Omega} \times \mathbf{r} \quad \text{on the sphere}, \quad r = a, \tag{2.8}$$

$$\mathbf{v} = 0 \quad \text{on the tube walls}, \quad R = R_o, \tag{2.9}$$

$$\mathbf{v} \sim \mathbf{v}^o \quad \text{as} \quad z \to \pm\infty, \tag{2.10}$$

$$p \sim \tfrac{1}{2}C + p^o \quad \text{as} \quad z \to -\infty, \tag{2.11}$$

$$p \sim -\tfrac{1}{2}C + p^o \quad \text{as} \quad z \to +\infty. \tag{2.12}$$

$\mathbf{U}_o$ is the translational velocity of the sphere centre $O$, $\mathbf{\Omega}$ is the angular velocity of the sphere, and $\mathbf{r}$ is the position vector measured with respect to the sphere centre. The values of the unknown vectors $\mathbf{U}_o$ and $\mathbf{\Omega}$ are ultimately to be determined by the condition that the sphere be neutrally buoyant. Conditions (2.10)–(2.12) are equivalent to the requirement that the disturbance to the Poiseuille flow vanish at large axial distances from the sphere. The positive constant $C$ in (2.11)–(2.12) reflects the *additional* pressure drop, $\Delta P^+$, above and beyond the Poiseuille pressure drop, attributable to the presence of the sphere in the flow; that is

$$C = \Delta P^+. \tag{2.13}$$

It is this quantity that we seek to calculate.

In view of the linearity of the governing equations and boundary conditions, the disturbance fields

$$\bar{\mathbf{v}} = \mathbf{v} - \mathbf{v}^o, \quad \bar{p} = p - p^o, \tag{2.14}$$

satisfy Stokes' equations,

$$\nabla^2 \bar{\mathbf{v}} = \mu_o^{-1} \nabla \bar{p}, \quad \nabla \cdot \bar{\mathbf{v}} = 0, \tag{2.15}$$

and the boundary conditions

$$\bar{\mathbf{v}} = \mathbf{U}_o + \mathbf{\Omega} \times \mathbf{r} - \mathbf{v}^o \quad \text{at} \quad r = a, \tag{2.16}$$

$$\bar{\mathbf{v}} = 0 \quad \text{at} \quad R = R_o, \tag{2.17}$$

$$\bar{\mathbf{v}} \sim 0 \quad \text{as} \quad z \to \pm\infty, \tag{2.18}$$

$$\bar{p} \sim \tfrac{1}{2}C \quad \text{as} \quad z \to -\infty, \tag{2.19}$$

$$\bar{p} \sim -\tfrac{1}{2}C \quad \text{as} \quad z \to +\infty. \tag{2.20}$$

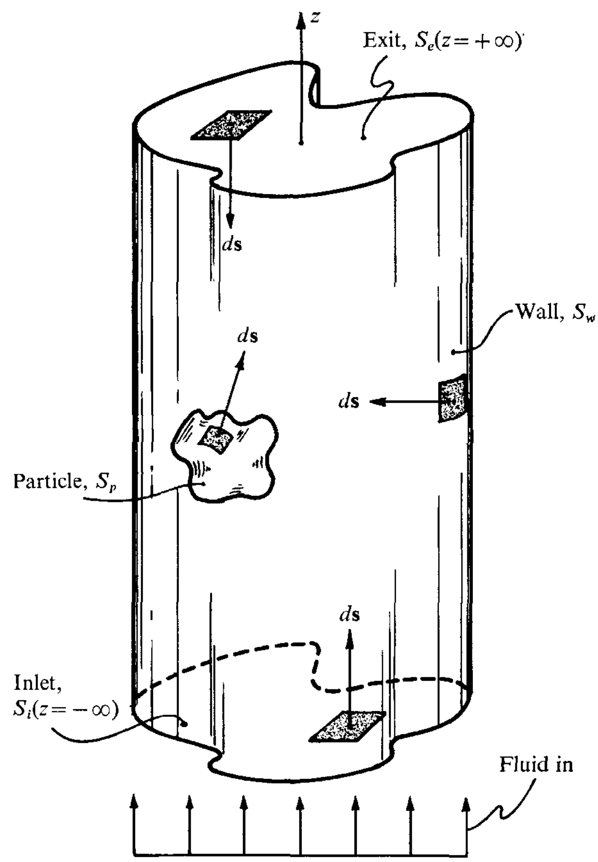


FIGURE 2. Particle in a duct of arbitrary cross-section.

A general formula for the additional pressure drop will now be developed. This formula holds for particles of arbitrary shape and cylindrical ducts of arbitrary (but constant) cross-section, as in figure 2. Consider the fluid bounded externally by the cylinder walls ($w$), the inlet ($i$) to the cylinder at $z = -\infty$, the exit ($e$) from the cylinder at $z = +\infty$, and bounded internally by the particle ($p$). To this region we may apply the following reciprocal theorem (Brenner 1963), valid for Stokes flows:

$$\oint_S d\mathbf{s} \cdot \mathbf{P} \cdot \mathbf{v}^o = \oint_S d\mathbf{s} \cdot \mathbf{P}^o \cdot \mathbf{v}, \tag{2.21}$$

where $S$ denotes the closed surface $S \equiv S_p + S_e + S_i + S_w$ bounding the fluid volume, and $\mathbf{P}$ is the pressure tensor, given generically by the formula

$$\mathbf{P} = -\mathbf{I}p + \mu_o[\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger]. \tag{2.22}$$

The directed element of surface area $d\mathbf{s}$ is taken to be drawn inward into the fluid volume, as depicted in figure 2. The contribution to these integrals from each of the four bounding surfaces will now be considered.

Since $\mathbf{v} = \mathbf{v}^o = 0$ on $S_w$, the side walls make no contribution to these integrals.

In view of the condition $\mathbf{v} = \mathbf{U}_o + \mathbf{\Omega} \times \mathbf{r}$ on $S_p$, the contribution of the particle to the right-hand integral is

$$\int_{S_p} d\mathbf{s} \cdot \mathbf{P}^o \cdot \mathbf{v} = \mathbf{U}_o \cdot \mathbf{F}^o + \mathbf{\Omega} \cdot \mathbf{T}_o^o,$$

where
$$\mathbf{F}^o = \int_{S_p} d\mathbf{s} \cdot \mathbf{P}^o, \quad \mathbf{T}_o^o = \int_{S_p} \mathbf{r} \times (d\mathbf{s} \cdot \mathbf{P}^o)$$

are the force and torque (about the particle centre $O$) exerted by the undisturbed flow on the region of space $V_p$ occupied by the particle. As $\mathbf{P}^o$ is non-singular in this region, the divergence theorem may be employed to convert these to volume integrals, yielding

$$\mathbf{F}^o = \int_{V_p} \nabla \cdot \mathbf{P}^o \, dV, \quad \mathbf{T}_o^o = \int_{V_p} \mathbf{r} \times (\nabla \cdot \mathbf{P}^o) \, dV.$$

But $\nabla \cdot \mathbf{P}^o = 0$ everywhere, giving $\mathbf{F}^o = 0$ and $\mathbf{T}_o^o = 0$. Consequently,

$$\int_{S_p} d\mathbf{s} \cdot \mathbf{P}^o \cdot \mathbf{v} = 0. \tag{2.23}$$

Upon collecting results we obtain

$$\int_{S_p} d\mathbf{s} \cdot \mathbf{P} \cdot \mathbf{v}^o + \int_{S_e+S_i} d\mathbf{s} \cdot \mathbf{P} \cdot \mathbf{v}^o = \int_{S_e+S_i} d\mathbf{s} \cdot \mathbf{P}^o \cdot \mathbf{v}. \tag{2.24}$$

At the inlet and exit, $z = \pm\infty$, we have that $\mathbf{v}^o = \hat{\mathbf{z}} v^o$ and $\mathbf{v} = \hat{\mathbf{z}} v \equiv \hat{\mathbf{z}} v^o$, since $v = v^o$ at $|z| = \infty$. Moreover, $d\mathbf{s}|_{S_e} = -\hat{\mathbf{z}}\, dA$ and $d\mathbf{s}|_{S_i} = \pm \hat{\mathbf{z}}\, dA$, where $dA$ is a scalar element of cross-sectional area on a plane $z =$ constant. Consequently, at the duct exit,

$$d\mathbf{s} \cdot \mathbf{P}^o \cdot \mathbf{v}|_{S_e} = -P_{zz}^o|_{S_e} v^o \, dA,$$

in which
$$P_{zz}^o|_{S_e} = -p_e^o + 2\mu_o \left.\frac{\partial v^o}{\partial z}\right|_{S_e} = -p_e^o,$$

where $p_e^o$ is the pressure at the exit in the absence of the particle. This makes

$$d\mathbf{s} \cdot \mathbf{P}^o \cdot \mathbf{v}|_{S_e} = p_e^o v^o \, dA. \tag{2.25}$$

Analogously, at the duct inlet,

$$d\mathbf{s} \cdot \mathbf{P}^o \cdot \mathbf{v}|_{S_i} = -p_i^o v^o \, dA. \tag{2.26}$$

Proceeding similarly for the other class of terms, we find at the exit that

$$d\mathbf{s} \cdot \mathbf{P} \cdot \mathbf{v}^o|_{S_e} = -P_{zz}|_{S_e} v^o \, dA,$$

where
$$P_{zz}|_{S_e} = -p_e + 2\mu_o \left.\frac{\partial v}{\partial z}\right|_{S_e} = -p_e.$$

We have here utilized the fact that $\partial v/\partial z|_{S_e} \sim \partial v^o/\partial z$ as $z \to \infty$, and that $v^o$ is independent of $z$.† Therefore,

$$d\mathbf{s}\cdot\mathbf{P}\cdot\mathbf{v}^o|_{S_e} = p_e v^o\,dA. \tag{2.27}$$

Likewise,
$$d\mathbf{s}\cdot\mathbf{P}\cdot\mathbf{v}^o|_{S_i} = -p_i v^o\,dA. \tag{2.28}$$

Substituting these results into (2.24) yields

$$\int_{S_p} d\mathbf{s}\cdot\mathbf{P}\cdot\mathbf{v}^o = \int_A (\overline{p}_i-\overline{p}_e)\,v^o\,dA,$$

in which
$$\overline{p}_i = p_i - p_i^o, \quad \overline{p}_e = p_e - p_e^o$$

are the disturbance pressures at the inlet and exit. Equations (2.19) and (2.20) show that $\overline{p}_i = \frac{1}{2}C$ and $\overline{p}_e = -\frac{1}{2}C$, respectively, so that from (2.13),

$$\overline{p}_i - \overline{p}_e = \Delta P^+.$$

Furthermore, from the definition of the mean velocity,

$$\int_A v^o\,dA = V_m A.$$

In this manner we obtain

$$\Delta P^+ V_m A = \int_{S_p} d\mathbf{s}\cdot\mathbf{P}\cdot\mathbf{v}^o. \tag{2.29}$$

The preceding relation expresses the additional pressure drop in terms of an appropriate integral over the surface of the particle. It applies to a particle of any shape and is valid for a cylindrical duct of arbitrary cross section, provided only that $\mathbf{v}^o$ refers to the undisturbed flow in that duct.

Evaluation of the integral appearing in (2.29) is a straightforward, but algebraically tedious, chore—especially when the particle is non-spherical. The requisite calculation can be greatly simplified by the following line of reasoning. Suppose that the field $(\mathbf{v}, p)$ defined by (2.7)–(2.12) has been determined. One can imagine this motion to be analytically continued outside the physical boundaries of the apparatus. By this device the motion may be regarded as extending to infinity. Let $S_\infty$ denote a spherical surface of indefinitely large radius containing the particle at its centre. In a purely formal manner, the general reciprocal theorem cited in (2.21) may be applied to the fluid (real and hypothetical) bounded externally by $S_\infty$ and internally by the particle. Consequently,

$$\int_{S_p+S_\infty} d\mathbf{s}\cdot\mathbf{P}\cdot\mathbf{v}^o = \int_{S_p+S_\infty} d\mathbf{s}\cdot\mathbf{P}^o\cdot\mathbf{v}.$$

This relation may be combined with (2.23) and (2.29) to yield

$$\Delta P^+ V_m A = \int_{S_\infty} d\mathbf{s}\cdot(\mathbf{P}^o\cdot\mathbf{v} - \mathbf{P}\cdot\mathbf{v}^o).$$

† The disturbance velocity created by a particle moving in a tube dies off exponentially rapidly (Sonshine & Brenner 1966) with axial distance $z$ for sufficiently large $|z|$, i.e. it is of order $\exp(-k|z|/R_o)$ where $k$ is a positive non-dimensional constant. Hence, no question exists as to the validity of setting $\partial v/\partial z = 0$ at $|z| = \infty$.

Since $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}^o$ and $\mathbf{P} = \bar{\mathbf{P}} + \mathbf{P}^o$, it follows that

$$\mathbf{P}^o \cdot \mathbf{v} - \mathbf{P} \cdot \mathbf{v}^o = \mathbf{P}^o \cdot \bar{\mathbf{v}} - \bar{\mathbf{P}} \cdot \mathbf{v}^o.$$

In addition, on the surface $S_\infty$, $d\mathbf{s} = -\hat{\mathbf{r}} r^2 \, d\Omega$ where $d\Omega = \sin\theta \, d\theta \, d\phi$ is an element of surface area on a unit sphere. Hence, we obtain

$$\Delta P^+ V_m A = \lim_{r\to\infty} \int_{S_\infty} r^2 (\bar{\mathbf{P}}_r \cdot \mathbf{v}^o - \mathbf{P}^o_r \cdot \bar{\mathbf{v}}) \, d\Omega, \tag{2.30}$$

where $\mathbf{P}_r = \hat{\mathbf{r}} \cdot \mathbf{P}$ is the stress vector on a spherical surface $r$ = constant.

The fields $\mathbf{v}^o$ and $\mathbf{P}^o$ are presumed to be known from the available solution of the elementary duct-flow problem. Equation (2.30) therefore enables a calculation of the additional pressure drop to be performed solely from a knowledge of the asymptotic behaviour of the perturbation field $(\bar{\mathbf{v}}, \bar{p})$ for large $r$. This calculation is significantly simpler than that required by (2.29) since it is now unnecessary to utilize the complete perturbation field in the calculation; rather, it suffices to retain only those terms leading to a contribution of $O(r^{-2})$ in the parenthetical term appearing in the integrand of (2.30). Furthermore, in the case of non-spherical particles, it is generally easier to integrate over the surface of a sphere than over the physical surface of the particle.

## 3. Additional pressure drop

The undisturbed velocity field, being analytic at all points within the cylinder, can be expanded in a polyadic Taylor series about the centre of the particle:

$$\mathbf{v}^o = \mathbf{v}^o_o + \boldsymbol{\omega}^o_o \times \mathbf{r} + \mathbf{r} \cdot \mathbf{S}^o_o + \tfrac{1}{2} \mathbf{rr} : \nabla\nabla \mathbf{v}^o. \tag{3.1}$$

For Poiseuille flow in a circular tube the expansion terminates with the term quadratic in $\mathbf{r}$. The double-dot notation conforms to the nesting convention. The subscript $o$ refers to the evaluation at the centre of the fluid volume presently occupied by the particle. In the above,

$$\boldsymbol{\omega}^o = \tfrac{1}{2} \nabla \times \mathbf{v}^o \tag{3.2}$$

and

$$\mathbf{S}^o = \tfrac{1}{2}[\nabla \mathbf{v}^o + (\nabla \mathbf{v}^0)^\dagger] \tag{3.3}$$

are, respectively, one-half the vorticity vector and rate of shear dyadic for the undisturbed flow. For the circular tube the coefficients in the Taylor expansion are, explicitly,

$$\mathbf{v}^o_o = \hat{\mathbf{z}} 2V_m(1-\beta^2), \quad \boldsymbol{\omega}^o_o = \hat{\mathbf{x}} 2R_o^{-1} V_m \beta, \tag{3.4), (3.5}$$

$$\mathbf{S}^o_o = (\hat{\mathbf{y}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{y}}) \, 2R_o^{-1} V_m \beta, \quad \nabla\nabla \mathbf{v}^o = 4R_o^{-2} V_m (\hat{\mathbf{z}}\hat{\mathbf{z}}\hat{\mathbf{z}} - \mathbf{I}\hat{\mathbf{z}}), \tag{3.6), (3.7}$$

in which

$$\beta = b/R_o \tag{3.8}$$

is the fractional distance of the centre of the particle from the tube axis. This function lies in the range $0 \leqslant \beta < 1$. For a spherical particle, the boundary condition (2.16) may be written as

$$\bar{\mathbf{v}} = (\mathbf{U}_o - \mathbf{v}^o_o) + a(\boldsymbol{\Omega} - \boldsymbol{\omega}^o_o) \times \hat{\mathbf{r}} - a\hat{\mathbf{r}} \cdot \mathbf{S}^o_o - \tfrac{1}{2} a^2 \hat{\mathbf{r}}\hat{\mathbf{r}} : \nabla\nabla \mathbf{v}^o \quad \text{at} \quad r = a. \tag{3.9}$$

The hydrodynamic force and torque on the particle arising from the complete state of motion $(\mathbf{v}, p)$ are

$$\mathbf{F} = \int_{S_p} d\mathbf{s} \cdot \mathbf{P}, \quad \mathbf{T}_o = \int_{S_p} \mathbf{r} \times (d\mathbf{s} \cdot \mathbf{P}). \tag{3.10}$$

A neutrally buoyant particle is one whose translational and angular velocities are such that $\mathbf{F} = 0$ and $\mathbf{T} = 0$, the torque then being independent of the choice of origin. Consider a spherical particle immersed in a Poiseuille flow. Suppose that the sphere rotates with angular velocity $\hat{\mathbf{x}}\Omega$ and that the sphere centre translates with velocity $\hat{\mathbf{z}}U$. The force and torque exerted by the fluid on the sphere are then (Brenner 1966*a*; Greenstein & Happel 1968)

$$\mathbf{F} = -\hat{\mathbf{z}}\, 6\pi\mu_o a \left[\frac{U - 2V_m(1-\beta^2) + \frac{4}{3}V_m\lambda^2}{1 - f(\beta)\lambda} + V_m O(\lambda^3)\right] \tag{3.11}$$

and

$$\mathbf{T}_o = -\hat{\mathbf{x}}\, 8\pi\mu_o a^2[a(\Omega - 2V_m\beta/R_o) + \{U - 2V_m(1-\beta^2)\} \times \{1 + g(\beta)\lambda\}\lambda^2 + V_m O(\lambda^4)], \tag{3.12}$$

where

$$\lambda = a/R_o. \tag{3.13}$$

The functions $f(\beta)$ and $g(\beta)$ are the wall-effect functions defined by Brenner & Happel (1958). Numerical values of these functions in the range $0 \leqslant \beta < 0{\cdot}9$ are tabulated by Famularo & Happel (1965) and, more accurately, by Greenstein & Happel (1968). An even more accurate tabulation over the entire $\beta$ range from 0 to 1 is given by Hirschfeld & Brenner (1970). Equations (3.11) and (3.12) are valid in the range $\lambda \ll 1-\beta$ [cf. equation (2.2)].

In the neutrally buoyant case it follows from (3.11) that the sphere centre translates with velocity

$$\mathbf{U}_o = \hat{\mathbf{z}}U, \tag{3.14}$$

where†

$$U = 2V_m(1-\beta^2) - \tfrac{4}{3}V_m\lambda^2 + V_m O(\lambda^3). \tag{3.15}$$

Equivalently,

$$\mathbf{U}_o = \mathbf{v}_o^o + \tfrac{1}{6}a^2\nabla^2\mathbf{v}^o + V_m O(\lambda^3), \tag{3.16}$$

where $\mathbf{v}_o^o$ is given by (3.4), and $\nabla^2\mathbf{v}^o$ is given by (2.6) with

$$\nabla p^o = -\hat{\mathbf{z}}\, 8\mu_o V_m/R_o^2 \tag{3.17}$$

as the Poiseuille's law pressure gradient. Upon substituting (3.15) into (3.12) and setting $\mathbf{T}_o = 0$, the angular velocity of a neutrally buoyant sphere is found to be

$$\mathbf{\Omega} = \hat{\mathbf{x}}\Omega, \tag{3.18}$$

where

$$\Omega = 2V_m\beta/R_o + a^{-1}g(\beta)V_m O(\lambda^4). \tag{3.19}$$

Equivalently,

$$a(\mathbf{\Omega} - \boldsymbol{\omega}_o^o) = g(\beta)V_m O(\lambda^4),$$

† The function $f(\beta)$ has the value $2{\cdot}104$ at $\beta = 0$, decreases to a minimum of $2{\cdot}044$ at $\beta = 0{\cdot}40$, and increases monotonically thereafter, ultimately attaining the asymptotic form $f(\beta) \sim \frac{9}{16}(1-\beta)^{-1}$ as $\beta \to 1$. Thus, in the latter limit, $f(\beta)\lambda \sim \frac{9}{16}\lambda/(1-\beta)$. Since the theory is confined to circumstances where $\lambda/(1-\beta) \ll 1$ [cf. equation (2.2)] it follows that

$$[1 - f(\beta)\lambda]V_m O(\lambda^3) = V_m O(\lambda^3)$$

for all values of $\beta$. This fact leads at once from (3.11) to (3.15).

where $\boldsymbol{\omega}_o^o$ is given by (3.5). The function $g(\beta)$ vanishes at $\beta = 0$ and increases monotonically, eventually attaining the asymptotic form† $g(\beta) = O(1-\beta)^{-1}$ as $\beta \to 1$. Since we have assumed that $\lambda/(1-\beta) \ll 1$, it follows that $g(\beta) = O(\lambda^{-1})$ for all $\beta$. Consequently,

$$a(\boldsymbol{\Omega} - \boldsymbol{\omega}_o^o) = V_m O(\lambda^3). \tag{3.20}$$

The error estimates in (3.16) and (3.20) prove crucial in the subsequent analysis.

In conjunction with (3.16) and (3.20) the boundary condition (3.9) becomes

$$\bar{\mathbf{v}} = -a\mathbf{P}_1(\hat{\mathbf{r}}) \cdot \mathbf{S}_o^o - \tfrac{1}{3}a^2\mathbf{P}_2(\hat{\mathbf{r}}) : \nabla\nabla\mathbf{v}^o + V_m O(\lambda^3) \quad \text{at} \quad r = a \tag{3.21}$$

for a neutrally buoyant sphere. Here,

$$\mathbf{P}_1(\hat{\mathbf{r}}) = \hat{\mathbf{r}}, \quad \mathbf{P}_2(\hat{\mathbf{r}}) = \tfrac{1}{2}(3\hat{\mathbf{r}}\hat{\mathbf{r}} - \mathbf{I}) \tag{3.22}$$

are the polyadic Legendre functions (Brenner 1964*b*; Ripps & Brenner 1967) of orders 1 and 2, respectively.

It proves convenient to decompose the perturbation field into a sum of two fields, as follows:

$$\bar{\mathbf{v}} = \mathbf{v}' + \mathbf{v}'', \quad \bar{p} = p' + p'', \tag{3.23}$$

in which the individual fields $(\mathbf{v}', p')$ and $(\mathbf{v}'', p'')$ each satisfy Stokes' equation and the continuity equation. The former field is required to satisfy the boundary conditions

$$\mathbf{v}' = -a\mathbf{P}_1(\hat{\mathbf{r}}) \cdot \mathbf{S}_o^o \quad \text{at} \quad r = a, \tag{3.24}$$

$$\mathbf{v}' = 0 \quad \text{at} \quad R = R_0, \qquad \mathbf{v}' \sim 0 \quad \text{as} \quad z \to \pm\infty, \qquad (3.25), (3.26)$$

while the latter is required to satisfy

$$\mathbf{v}'' = -\tfrac{1}{3}a^2\mathbf{P}_2(\hat{\mathbf{r}}) : \nabla\nabla\mathbf{v}^o + V_m O(\lambda^3) \quad \text{at} \quad r = a, \tag{3.27}$$

$$\mathbf{v}'' = 0 \quad \text{at} \quad R = R_o, \qquad \mathbf{v}'' \sim 0 \quad \text{as} \quad z \to \pm\infty. \qquad (3.28), (3.29)$$

It is convenient to refer to the primed and double-primed fields as the 'shear' and 'quadratic' contributions, respectively. Since the pressure drop formula (2.30) is linear in $(\bar{\mathbf{v}}, \bar{p})$ the respective contributions of these fields to the additional pressure drop may be separately treated and the results subsequently added.

*Shear contribution*

Calculation of $(\mathbf{v}', p')$ requires that we solve the Stokes and continuity equations so as to satisfy the boundary conditions (3.24)–(3.26) for small $\lambda$. Such problems can be solved to any order in $\lambda$ by the method of reflexions (Brenner & Happel 1958), the method of Haberman & Sayre (1958), or the method of matched asymptotic expansions (Cox & Brenner 1967). Employing the first-mentioned technique, we write

$$\mathbf{v}' = \mathbf{v}_1' + \mathbf{v}_2' + \mathbf{v}_3' + \ldots, \quad p' = p_1' + p_2' + p_3' + \ldots, \tag{3.30}$$

where each separate field, $(\mathbf{v}_j', p_j')$, satisfies the equations of motion. The following boundary conditions are imposed on the individual fields:

$$\mathbf{v}_1' = \left\{ \begin{matrix} -a\mathbf{P}_1(\hat{\mathbf{r}}) \cdot \mathbf{S}_o^o & \text{at} & r = a, \\ 0 & \text{as} & r \to \infty, \end{matrix} \right\} \tag{3.31}$$

† That this is so can be seen from the numerical tabulations (Greenstein & Happel 1968; Hirschfeld & Brenner 1971) of $g(\beta)$ *vs.* $\beta$, which clearly show that $(1-\beta)g(\beta)$ approaches a small, finite constant as $\beta \to 1$.

$$\mathbf{v}_2' = \begin{cases} -\mathbf{v}_1' & \text{at} \quad R = R_o, \\ 0 & \text{as} \quad z \to \pm\infty, \end{cases} \tag{3.32}$$

$$\mathbf{v}_3' = \begin{cases} -\mathbf{v}_2' & \text{at} \quad r = a, \\ 0 & \text{as} \quad r \to \infty, \end{cases} \tag{3.33}$$

etc. The odd-numbered fields constitute the 'reflexion' of the preceding, even-numbered, velocity fields from the surface of the sphere. Conversely, the even-numbered fields correspond to the reflexion of the preceding, odd-numbered, fields from the cylinder surface. The even fields are required to be finite at all points within the cylinder, including the region of space presently occupied by the sphere. In contrast, the odd fields will possess singularities in the interior of the sphere.

The first reflexion, satisfying (3.31), is easily obtained via the general methods outlined by Brenner (1964*a*). A general solution of the Stokes' and continuity equations which vanishes at infinity is

$$\mathbf{v} = \sum_{n=1}^{\infty} \left[ \nabla \times (\mathbf{r}\chi_{-(n+1)}) + \nabla \varphi_{-(n+1)} - \frac{(n-2)}{2n(2n-1)\mu_o} r^2 \nabla p_{-(n+1)} + \mathbf{r} \frac{(n+1)}{n(2n-1)\mu_o} p_{-(n+1)} \right], \tag{3.34}$$

$$p = \sum_{n=1}^{\infty} p_{-(n+1)}, \tag{3.35}$$

in which $\chi_{-(n+1)}, \varphi_{-(n+1)}, p_{-(n+1)}$ are solid spherical harmonics of order $-(n+1)$. The Einstein (1905, 1911) field $(\mathbf{v}_1', p_1')$ satisfying (3.31) is given by the above with the values†

$$p_{-3} = -\tfrac{10}{3}\mu_o (a/r)^3 \mathbf{P}_2(\hat{\mathbf{r}}) : \mathbf{S}_o^o, \tag{3.36}$$

$$\varphi_{-3} = -\tfrac{1}{3}a^2 (a/r)^3 \mathbf{P}_2(\hat{\mathbf{r}}) : \mathbf{S}_o^o, \tag{3.37}$$

all the other spherical harmonics being zero. Owing to the incompressibility of the fluid, $\mathbf{I} : \mathbf{S}_o^o = 0$. More explicitly, substituting the value of $\mathbf{S}_o^o$ from (3.6) yields

$$p_{-3} = -\tfrac{20}{3}\lambda\beta a^{-1} V_m (a/r)^3 P_2^1(\cos\theta) \sin\phi, \tag{3.38}$$

$$\varphi_{-3} = -\tfrac{2}{3}\lambda\beta a V_m (a/r)^3 P_2^1(\cos\theta) \cos\phi, \tag{3.39}$$

in which $P_n^m(\cos\theta)$ is an ordinary Legendre function; in particular,

$$P_2^1(\cos\theta) = 3\sin\theta\cos\theta.$$

The dominant term governing the asymptotic behaviour of $\mathbf{v}_1'$ as $r \to \infty$ is a term of $O(r^{-2})$ arising from the $p_{-3}$ harmonic; that is,

$$\mathbf{v}_1' = -\tfrac{10}{3} V_m \lambda\beta (\mathbf{r}/a)(a/r)^3 P_2^1(\cos\theta)\sin\phi + V_m\lambda\beta O(a/r)^4 \quad \text{as} \quad r \to \infty. \tag{3.40}$$

The term displayed explicitly is of the form $V_m \lambda\beta O(a/r)^2$.

† Note that, in general, terms of the form $r^{-(n+1)}\mathbf{P}_n(\hat{\mathbf{r}})$ are polyadic solid spherical harmonics of order $-(n+1)$. Thus, if $\mathbf{A}_n$ is any $n$-adic constant, the scalar field

$$r^{-(n+1)}\mathbf{A}_n \langle n \rangle \mathbf{P}_n$$

is an ordinary solid spherical harmonic. (The symbol $\langle n \rangle$ denotes $n$-dot multiplication.) Indeed, $\mathbf{A}_n \langle n \rangle \mathbf{P}_n$ is an ordinary scalar surface spherical harmonic of degree $n$.

On the cylinder wall we have that $r = O(R_o)$. Consequently, for small $\lambda$,

$$\mathbf{v}_1' = V_m \beta O(\lambda^3) \quad \text{at} \quad R = R_o.$$

It thus follows from the boundary conditions (3.32) defining $\mathbf{v}_2'$ that

$$\mathbf{v}_2' = V_m \beta O(\lambda^3) \quad \text{at} \quad R = R_o$$

in so far as dominant terms in $\lambda$ are concerned. Since $\mathbf{v}_2'$ is regular at all points in the interior of the cylinder, this field will be of the form

$$\mathbf{v}_2' = V_m \lambda^3 \beta \hat{\mathbf{f}}(\hat{\mathbf{R}})$$

in which $\hat{\mathbf{R}}$ denotes a general position vector made dimensionless with the tube radius $R_o$, and $\hat{\mathbf{f}}(\hat{\mathbf{R}})$ is a dimensionless vector function which is of $O(1)$ everywhere in the tube. The latter will generally depend upon $\beta$, but in a non-singular manner. Expansion in Taylor series about the sphere centre thereby yields

$$\mathbf{v}_2' = V_m \lambda^3 \beta [\hat{\mathbf{f}}_o + \tfrac{1}{2}(\nabla \times \hat{\mathbf{f}})_o \times \mathbf{r} + \mathbf{r} \cdot (\nabla \hat{\mathbf{f}})_o^{\text{sym}} + \tfrac{1}{2}\mathbf{rr} \colon (\nabla\nabla\hat{\mathbf{f}})_o + \ldots],$$

in which $(\nabla \hat{\mathbf{f}})^{\text{sym}} = \frac{1}{2}[\nabla \hat{\mathbf{f}} + (\nabla \hat{\mathbf{f}})^\dagger]$ is the symmetric part of the dyadic $\nabla \hat{\mathbf{f}}$. As before, the subscript $o$ refers to evaluation of the function to which it is appended at the sphere centre. In terms of the dimensionless gradient operator $\hat{\nabla} = R_o \nabla$, the above expansion may be written as

$$\mathbf{v}_2' = V_m \lambda^3 \beta \left[\hat{\mathbf{f}}_o + \frac{1}{2}\frac{r}{R_o}(\hat{\nabla} \times \hat{\mathbf{f}})_o \times \hat{\mathbf{r}} + \frac{r}{R_o}\hat{\mathbf{r}} \cdot (\hat{\nabla}\hat{\mathbf{f}})_o^{\text{sym}} + \frac{1}{2}\left(\frac{r}{R_o}\right)^2 \hat{\mathbf{r}}\hat{\mathbf{r}} \colon (\hat{\nabla}\hat{\nabla}\hat{\mathbf{f}})_o + \ldots\right]. \tag{3.41}$$

Upon putting $r = a$ in the above, it follows from (3.33) that $\mathbf{v}_3'$ is required to vanish at $r = \infty$ and to satisfy the following boundary condition on the sphere surface:

$$\mathbf{v}_3' = -V_m \lambda^3 \beta [\hat{\mathbf{f}}_o + \tfrac{1}{2}\lambda(\hat{\nabla} \times \hat{\mathbf{f}})_o \times \hat{\mathbf{r}} + \lambda \hat{\mathbf{r}} \cdot (\hat{\nabla}\hat{\mathbf{f}})_o^{\text{sym}} + \tfrac{1}{2}\lambda^2 \hat{\mathbf{r}}\hat{\mathbf{r}} \colon (\hat{\nabla}\hat{\nabla}\hat{\mathbf{f}})_o + \ldots] \quad \text{at} \quad r = a. \tag{3.42}$$

The constant polyadics $\hat{\mathbf{f}}_o$, $(\hat{\nabla} \times \hat{\mathbf{f}})_o$, $(\hat{\nabla}\hat{\mathbf{f}})_o^{\text{sym}}$, $(\hat{\nabla}\hat{\nabla}\hat{\mathbf{f}})_o, \ldots$, are all of $O(1)$, as are the unit vectors $\hat{\mathbf{r}}$.

The even-numbered fields in (3.30) can exert neither a force nor torque on the particle since they are non-singular in the space occupied by the particle (cf. the analogous arguments preceding (2.23) in regard to the non-singular field $\mathbf{v}^o$). Consequently,

$$\mathbf{F}' = \mathbf{F}_1' + \mathbf{F}_3' + \ldots, \quad \mathbf{T}_o' = (\mathbf{T}_1')_o + (\mathbf{T}_3')_o + \ldots. \tag{3.43}$$

The force and torque on the sphere associated with (3.34)–(3.35) are, in general (Brenner 1964*a*),

$$\mathbf{F} = -4\pi \nabla(r^3 p_{-2}), \quad \mathbf{T}_o = -8\pi\mu_o \nabla(r^3 \chi_{-2}). \tag{3.44}$$

As regards the field $\mathbf{v}_1'$, we have *inter alia* that $p_{-2} = \chi_{-2} = 0$, since only the $p_{-3}$ and $\varphi_{-3}$ harmonics are non-zero. Hence,

$$\mathbf{F}_1' = 0, \quad (\mathbf{T}_1')_o = 0. \tag{3.45}$$

From (3.43) the conditions of neutral buoyancy, $\mathbf{F}' = 0$ and $\mathbf{T}_0' = 0$, thereby require that

$$\mathbf{F}_3' + \mathbf{F}_5' + \ldots = 0, \quad (\mathbf{T}_3')_o + (\mathbf{T}_5')_o + \ldots = 0. \tag{3.46}$$

By Faxén's laws (Brenner 1966*b*) we have that

$$\mathbf{F}_3' = 6\pi\mu_o a[(\mathbf{v}_2')_o + \tfrac{1}{6}a^2(\nabla^2\mathbf{v}_2')_o] \tag{3.47}$$

and

$$(\mathbf{T}_3')_o = 4\pi\mu_o a^3(\nabla\times\mathbf{v}_2')_o, \tag{3.48}$$

whence, from (3.41),

$$\mathbf{F}_3' = 6\pi\mu_o a V_m \lambda^3\beta[\hat{\mathbf{f}}_o + \tfrac{1}{6}\lambda^2(\hat{\nabla}^2\hat{\mathbf{f}})_o] \tag{3.49}$$

and

$$(\mathbf{T}_3')_o = 4\pi\mu_o a^3 V_m \lambda^4\beta(\hat{\nabla}\times\hat{\mathbf{f}})_o. \tag{3.50}$$

As will be demonstrated *a posteriori* in the following paragraph, the condition of neutral buoyancy requires that

$$\hat{\mathbf{f}}_o = -\tfrac{1}{6}\lambda^2(\hat{\nabla}^2\hat{\mathbf{f}})_o + O(\lambda^3), \quad (\hat{\nabla}\times\hat{\mathbf{f}})_o = O(\lambda^3). \tag{3.51}$$

Upon substituting the latter relations into (3.42), the boundary condition satisfied by $\mathbf{v}_3'$ on the sphere becomes

$$\mathbf{v}_3' = -V_m\lambda^4\beta\mathbf{P}_1(\hat{\mathbf{r}})\cdot(\hat{\nabla}\hat{\mathbf{f}})_o^{\text{sym}} + V_m\beta O(\lambda^5) \quad \text{at} \quad r = a. \tag{3.52}$$

Comparison with the completely analogous problem defined by (3.31) leads immediately to the conclusion that

$$\mathbf{v}_3' = V_m\beta O\{\lambda^4(a/r)^2\} \quad \text{as} \quad r\to\infty, \tag{3.53}$$

$$p_3' = \mu_o a^{-1}V_m\beta O\{\lambda^4(a/r)^3\} \quad \text{as} \quad r\to\infty. \tag{3.54}$$

Proceeding as before, it can readily be demonstrated that $\mathbf{v}_4' = V_m\beta O(\lambda^6)$ at $R = R_o$, and eventually that

$$\mathbf{F}_5' = 6\pi\mu_o a V_m\lambda^6\beta[\hat{\mathbf{g}}_o + O(\lambda)], \tag{3.55}$$

$$(\mathbf{T}_5')_o = 4\pi\mu_o a^3 V_m\lambda^7\beta[(\hat{\nabla}\times\hat{\mathbf{g}})_o + O(\lambda)], \tag{3.56}$$

where $\hat{\mathbf{g}}$ is a dimensionless vector function of $O(1)$ which bears the same relation to $\mathbf{v}_4'$ as $\hat{\mathbf{f}}$ bears to $\mathbf{v}_2'$. The terms of $O(\lambda)$ in square brackets stem from the neglected term of $O(\lambda^5)$ in (3.52). Upon adding the latter relations to (3.49) and (3.50), respectively, and taking account of (3.46), we obtain

$$\hat{\mathbf{f}}_o + \tfrac{1}{6}\lambda^2(\hat{\nabla}^2\mathbf{f})_o + \lambda^3\hat{\mathbf{g}}_o + o(\lambda^3) = 0$$

and

$$(\hat{\nabla}\times\hat{\mathbf{f}})_o + \lambda^3(\hat{\nabla}\times\hat{\mathbf{g}})_o + o(\lambda^3) = 0.$$

Since $\hat{\mathbf{g}}_o$ and $(\hat{\nabla}\times\hat{\mathbf{g}})_o$ are of $O(1)$, (3.51) (and, hence, (3.53) and (3.54)) are confirmed.

The even-numbered velocity and pressure fields decay exponentially rapidly as $r\to\infty$ (cf. the first footnote p. 646). Accordingly, we conclude that the *exact* solution of the 'shear' boundary-value problem defined by (3.24)–(3.26) will be of the asymptotic form

$$\mathbf{v}' = \mathbf{v}_1' + V_m\beta O\{\lambda^4(a/r)^2\} \quad \text{as} \quad r\to\infty, \tag{3.57}$$

$$p' = p_1' + \mu_o a^{-1}V_m\beta O\{\lambda^4(a/r)^3\} \quad \text{as} \quad r\to\infty, \tag{3.58}$$

in which $(\mathbf{v}_1', p_1')$ are given by (3.34)–(3.35) with the values of the harmonic functions cited in (3.36)–(3.37). The corresponding pressure tensor is

$$\mathbf{P}' = \mathbf{P}_1' + \mu_o a^{-1}V_m\beta O\{\lambda^4(a/r)^3\} \quad \text{as} \quad r\to\infty.$$

The comparable stress vector,

$$\mathbf{P}'_r = (\mathbf{P}'_1)_r + \mu_o a^{-1} V_m \beta O\{\lambda^4 (a/r)^3\} \quad \text{as} \quad r \to \infty, \tag{3.59}$$

may be computed from the general formula (Brenner 1964*a*)†

$$\mathbf{P}_r = \frac{\mu_o}{r} \sum_{n=1}^{\infty} \left[ -(n+2)\nabla \times (\mathbf{r}\chi_{-(n+1)}) - 2(n+2)\nabla \varphi_{-(n+1)} - \frac{(2n^2+1)}{n(2n-1)\mu_o} \mathbf{r} p_{-(n+1)} + \frac{(n+1)(n-1)}{n(2n-1)\mu_o} r^2 \nabla p_{-(n+1)} \right], \tag{3.60}$$

which pertains to the general solution (3.34)–(3.35). The value of $(\mathbf{P}'_1)_r$ may be obtained from this expression by inserting the values for $p_{-3}$ and $\varphi_{-3}$ given by (3.36)–(3.37) and putting all the other solid spherical harmonics equal to zero.

Upon substituting into (2.30) and performing the requisite integrations,‡ the contribution of the 'shear' field $(\mathbf{v}', p')$ to the additional pressure drop is found to be

$$(\Delta P^+)' V_m A = \tfrac{160}{3} \pi \mu_o R_o V_m^2 \beta^2 \lambda^3 [1 + O(\lambda^3)]. \tag{3.61}$$

*Quadratic contribution*

Attention is now directed to the computation of the 'quadratic' contribution $(\mathbf{v}'', p'')$ defined by the boundary conditions (3.27)–(3.29). Employing the expression for $\nabla\nabla\mathbf{v}^o$ given by (3.7), noting that $\mathbf{P}_2(\hat{\mathbf{r}}) : \mathbf{I} = 0$ and that

$$\mathbf{P}_2(\hat{\mathbf{r}}) \vdots \hat{\mathbf{z}}\hat{\mathbf{z}} = P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1)$$

(where $P_n$ is the Legendre polynomial of order $n$), we find that $\mathbf{v}''$ satisfies the following boundary condition on the sphere:

$$\mathbf{v}'' = -\hat{\mathbf{z}}\tfrac{4}{3} V_m \lambda^2 P_2(\cos\theta) + V_m O(\lambda^3) \quad \text{at} \quad r = a. \tag{3.62}$$

As in the shear case, this boundary-value problem may be solved for small $\lambda$ by the method of reflexions. Define

$$\mathbf{v}'' = \mathbf{v}''_1 + \mathbf{v}''_2 + \mathbf{v}''_3 + \ldots + V_m O(\lambda^3), \quad p'' = p''_1 + p''_2 + p''_3 + \ldots + \mu_o a^{-1} V_m O(\lambda^3), \tag{3.63}$$

where $(\mathbf{v}''_j, p''_j)$ satisfy the equations of motion and the boundary conditions

$$\mathbf{v}''_1 = \left\{ \begin{array}{ll} -\hat{\mathbf{z}}\tfrac{4}{3} V_m \lambda^2 P_2(\cos\theta) & \text{at} \quad r = a, \\ 0 & \text{as} \quad r \to \infty, \end{array} \right\} \tag{3.64}$$

† Alternatively, it may be computed from the relation

$$\mathbf{P}_r = -\hat{\mathbf{r}} p + \mu_o \left( \frac{\partial \mathbf{v}}{\partial r} - \frac{\mathbf{v}}{r} \right) + \frac{\mu_o}{r} \nabla(\mathbf{r} \cdot \mathbf{v})$$

valid for an incompressible Newtonian fluid.

‡ Employing the integration methods of Schowalter, Chaffey & Brenner (1968), we obtain

$$(\Delta P^+)' V_m A = -\tfrac{8}{3}\pi \nabla\nabla(r^5 p_{-3}) : \mathbf{S}^o_o + o(\lambda^3)$$

in general. For the case of a spherical particle in an arbitrary field of flow, we find upon substituting from (3.36), that

$$(\Delta P^+)' V_m A = \tfrac{20}{3} \pi \mu_o \mathbf{S}^o_o : \mathbf{S}^o_o + o(\lambda^3).$$

Note that the $\varphi_{-3}$ harmonic makes no contribution to the pressure drop, as follows immediately from the fact that its contributions to $\mathbf{v}'$ and $\mathbf{P}'_r$ are of orders $r^{-4}$ and $r^{-5}$, respectively.

$$\mathbf{v}_2'' = \begin{cases} -\mathbf{v}_1'' & \text{at} \quad R = R_o, \\ 0 & \text{as} \quad z \to \pm\infty, \end{cases} \tag{3.65}$$

$$\mathbf{v}_3'' = \begin{cases} -\mathbf{v}_2'' & \text{at} \quad r = a, \\ 0 & \text{as} \quad r \to \infty, \end{cases} \tag{3.66}$$

etc.

The solution of the first-order equations (3.64) may be expressed in terms of the general solution (3.34)–(3.35) by eliminating all harmonics except for the following:

$$\varphi_{-2} = \tfrac{4}{15} a V_m \lambda^2 (a/r)^2 P_1(\cos\theta), \tag{3.67}$$

$$\varphi_{-4} = -\tfrac{1}{2} a V_m \lambda^2 (a/r)^4 P_3(\cos\theta), \tag{3.68}$$

$$p_{-4} = -7\mu_o a^{-1} V_m \lambda^2 (a/r)^4 P_3(\cos\theta), \tag{3.69}$$

where $P_1(\cos\theta) = \cos\theta$ and $P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$. The dominant term in this velocity field as $r \to \infty$ is of the form $\mathbf{v}_1'' = V_m \lambda^2 O(a/r)^3$. Since $r = O(R_o)$ on the cylinder wall, this leads, by the same reasoning as in the shear case, to the conclusion that $\mathbf{v}_2'' = V_m O(\lambda^5)$. This term is already of larger order in $\lambda$ than the error term $V_m O(\lambda^3)$ appearing in (3.63), and hence may be neglected.

Since the sphere experiences no force, there can be no term in $\mathbf{v}''$ of $O(r^{-1})$;† only a term of $O(r^{-n})$ $(n \geqslant 2)$ is possible. It may be concluded, therefore, that

$$\mathbf{v}'' = \mathbf{v}_1'' + V_m O\{\lambda^3 (a/r)^2\} \quad \text{as} \quad r \to \infty, \tag{3.70}$$

$$p'' = p_1'' + \mu_o a^{-1} V_m O\{\lambda^3 (a/r)^3\} \quad \text{as} \quad r \to \infty. \tag{3.71}$$

Utilizing (2.30) and proceeding as before, the contribution of $(\mathbf{v}'', p'')$ to the additional pressure drop is ultimately found to be

$$(\Delta P^+)'' V_m A = 16\pi\mu_o R_o V_m^2 \lambda^5 [1 + O(\lambda)]. \tag{3.72}$$

## 4. Results for a neutrally buoyant sphere in a circular tube

Equations (3.61) and (3.72) may be added to obtain the additional pressure drop:

$$\Delta P^+ V_m A = \tfrac{160}{3}\pi\mu_o R_o V_m^2 \beta^2 \lambda^3 + 16\pi\mu_o R_o V_m^2 \lambda^5 + \mu_o R_o V_m^2 O(\lambda^6), \tag{4.1}$$

where $A = \pi R_o^2$. The error term in this expression is a function of $\beta$; that is, it is of the form $\mu_o R_o V_m^2 F(\beta) O(\lambda^6)$ where $F(\beta)$ is a non-dimensional function of $O(1)$. Equation (4.1) is one of the principal results of this paper. The term $\Delta P^+ V_m A$ represents the additional mechanical energy dissipated in the duct due to the presence of the sphere, above and beyond the Poiseuille dissipation.

The leading term in (4.1) is a special case of a more general formula for the additional rate of dissipation $\dot{D}^+$, arising from the presence of a neutrally buoyant sphere in an *arbitrary* field of flow $\mathbf{v}^o$ (Brenner 1959, 1962, 1966*a*),

$$\dot{D}^+ = \tfrac{5}{2} v_p \Phi_o^o + o(a/l)^3, \tag{4.2}$$

† The only non-trivial term in (3.34) that can lead to a term of $O(r^{-1})$ in the velocity field is the $p_{-2}$ harmonic. However, since the force on the neutrally buoyant sphere is zero, (3.44) shows that we must have $p_{-2} = 0$.

where $v_p = \frac{4}{3}\pi a^3$ is the volume of the sphere, $l$ is a characteristic apparatus dimension, and

$$\Phi_o^o = 2\mu_o \mathbf{S}_o^o : \mathbf{S}_o^o \tag{4.3}$$

is the local rate of mechanical energy dissipation for the undisturbed flow evaluated at the centre of the space occupied by the sphere. For Poiseuille flow, $\mathbf{S}_o^o$ is given by (3.6), whence $\Phi_o^o = 16\mu_o V_m^2 \beta^2/R_o^2$. Setting $D^+ = \Delta P^+ V_m A$ and putting $l = R_o$ thereby yields the leading term of (4.1). The factor of $\frac{5}{2}$ in (4.2) is the Einstein (1905, 1911) factor in the formula $\mu = \mu_o(1+\frac{5}{2}\phi)$ for the apparent viscosity of a uniform suspension of spheres.

Equation (4.1) may be expressed in the alternative form

$$\Delta P^+ R_o/\mu_o V_m = \tfrac{160}{3}\beta^2\lambda^3 + 16\lambda^5 + O(\lambda^6). \tag{4.4}$$

This formula applies to the case where $\lambda \ll 1-\beta$ [cf. equation (2.2)]. This criterion stems from the failure of the method of reflexions to converge when the particle is too near the wall. This restriction does not imply that $\beta$ cannot approach unity. For example, if $\lambda = 10^{-4}$ and $\beta = 0{\cdot}99$, then $\lambda/(1-\beta) = 0{\cdot}01 \ll 1$, and the criterion is well satisfied.

The error estimate in (4.4) applies only to cases where $\beta \neq 0$. The error is much smaller for the concentric case, $\beta = 0$. The difference between the concentric and eccentric cases stems from four sources: (i) The work of Haberman & Sayre (1958) shows† that, for $\beta = 0$, (3.15) is valid to $O(\lambda^5)$ rather than $O(\lambda^3)$. Though the term of $O(\lambda^3)$ in (3.11) has never been calculated explicitly for $\beta \neq 0$, it can be demonstrated to be non-zero. (ii) In the concentric case, the local vorticity and the angular velocity of the sphere are identically zero. The relation $\mathbf{\Omega} - \boldsymbol{\omega}_o^o = 0$ is then valid to all orders in $\lambda$, rather than only to $O(\lambda^3)$ as in (3.20). (iii) When $\beta = 0$ the harmonic functions $p_{-3}$ and $\varphi_{-3}$ in (3.38)–(3.39) vanish identically, owing to the absence of local shear at the tube axis. Consequently, the disturbance velocity $\bar{\mathbf{v}}$ is no longer of $O\{\lambda(a/r)^2\}$ as in (3.40), but is rather of $O\{\lambda^2(a/r)^3\}$, as in the leading term of (3.70). (iv) Symmetry arguments show that the 'shear' terms $(\hat{\nabla}\times\hat{\mathbf{f}})_o$ and $(\hat{\nabla}\hat{\mathbf{f}})_o^{\mathrm{sym}}$ in (3.41) are necessarily zero for a concentrically located sphere. Since $\hat{\mathbf{f}}_o = O(\lambda^2)$, then, to terms of lowest order in $\lambda$, the Taylor expansion (3.41) begins with a term of $O(r^2)$ rather than one of $O(r)$, as was formerly true for the case $\beta \neq 0$.

When the error analysis is appropriately altered to take account of the very special circumstances pertaining to the case $\beta = 0$, it is easy to show that the error term in (4.4) becomes $O(\lambda^{10})$. The additional pressure drop for the symmetric case, $\beta = 0$, is therefore of the form

$$\Delta P^+ R_o/\mu_o V_m = 16\lambda^5 + O(\lambda^{10}). \tag{4.5}$$

This result was obtained independently by Hochmuth & Sutera (1970) from a detailed solution of the corresponding axisymmetric boundary-value problem. Their analysis gives explicitly the term of $O(\lambda^{10})$, as well as several other higher-order terms in the expansion. Moreover, they cite experimental evidence in support of this relation.

† Explicitly, the analogue of (3.11) for $\beta = 0$ is

$$\mathbf{F} = -\hat{\mathbf{z}}6\pi\mu_o a\left[\frac{U - 2V_m + \frac{4}{3}V_m\lambda^2 + O(\lambda^5)}{1 - 2{\cdot}104\lambda + 2{\cdot}09\lambda^3 - 1{\cdot}71\lambda^5}\right].$$

Lubrication-theory-like calculations have been utilized to obtain comparable expressions for the translational and angular velocities of the sphere, and the additional pressure drop, for the case of a large, closely-fitting sphere, $\lambda \to 1$. This has been done for both the concentric case (Hochmuth & Sutera 1970) and the eccentric case (Bungay & Brenner 1970).

The eccentricity at which the two terms in (4.4) become comparable occurs at the value $\beta = \lambda(3/10)^{\frac{1}{2}}$. For $\lambda = 0{\cdot}1$ this corresponds to a value of $\beta = 0{\cdot}055$. Use of the concentric-sphere formula (4.5) for a particle which appears visually to be centred at the tube axis may therefore lead to appreciable errors, even for a relatively small degree of eccentricity.

Small departures from a precise state of neutral buoyancy may also result in appreciable errors. Consider a spherical particle in a vertical tube. The additional pressure drop force due solely to the drag on the particle is (Brenner & Happel 1958; Brenner 1962) $\Delta P_D^+ A = 2(1-\beta^2) D$, in which the drag force $D$ is the weight of the particle corrected for the buoyancy of the fluid: $D = \frac{4}{3}\pi a^3 \Delta\rho g$, where $\Delta\rho$ is the density difference between the particle and the fluid and $g$ is the acceleration of gravity. This makes

$$\Delta P_D^+ = \tfrac{8}{3}\lambda^3(1-\beta^2)\,\Delta\rho g R_o. \tag{4.6}$$

Consequently,

$$\frac{\Delta P_D^+}{\Delta P^+} = \frac{1-\beta^2}{\frac{10}{3}\beta^2+\lambda^2}\,\frac{4\Delta\rho g}{3(\Delta p^o/L)}, \tag{4.7}$$

in which

$$\Delta p^o/L = 8\mu_o V_m/R_o^2 \tag{4.8}$$

is the Poiseuille pressure gradient in the absence of the particle. For $\beta \neq 0$, (4.7) shows that the error incurred in the additional pressure drop by ignoring the density difference is of the order of $\Delta\rho g(\Delta p^o/L)^{-1}$. This may be very large if the pressure gradient is very small. The situation is much worse if $\beta = 0$; for then the error is essentially $1/\lambda^2$ times as large. And this increases without bound as $\lambda \to 0$.

## 5. Non-circular ducts

The pressure drop formula [cf. equations (4.2)–(4.3)]

$$\Delta P^+ V_m A = \tfrac{20}{3}\pi\mu_o a^3 \mathbf{S}_o^o : \mathbf{S}_o^o + o(\lambda^3) \tag{5.1}$$

is valid to the indicated order in $\lambda$ for a neutrally-buoyant spherical particle in a cylindrical duct of any cross-section, provided only that one inserts the value of $\mathbf{S}^o$ appropriate to that duct. Since the undisturbed flow $\mathbf{v}^o$ in a cylinder of constant cross-section is unidirectional we may write $\mathbf{v}^o = \hat{\mathbf{z}} v^o$, where $v^o \equiv v^o(X, Y)$. The general formula (5.1) reduces in this case to

$$\Delta P^+ V_m A = \tfrac{10}{3}\pi a^3 (\nabla v^o)_o^2 + o(\lambda^3), \tag{5.2}$$

in which $(\nabla v^o)^2 = (\nabla v^o)\cdot(\nabla v^o)$.

Consider, for example, flow in an elliptic conduit of semi-axes $R_1$ and $R_2$. For this case (Lamb 1932, p. 587),

$$v^o = 2V_m\left(1-\frac{X^2}{R_1^2}-\frac{Y^2}{R_2^2}\right), \tag{5.3}$$

where $X$ and $Y$ are measured from the centre of the ellipse. If the lateral position of the sphere centre corresponds to the co-ordinates $X = b_1$, $Y = b_2$ this gives

$$\Delta P^+ V_m A = \tfrac{160}{3}\pi\mu_o a^3 V_m^2 \left(\frac{b_1^2}{R_1^4}+\frac{b_2^2}{R_2^4}\right) + o(\lambda^3). \tag{5.4}$$

The cross-sectional area is $A = \pi R_1 R_2$. Consequently,

$$\Delta P^+(R_1+R_2)/\mu_o V_m = \tfrac{160}{3}\pi(\lambda_1+\lambda_2)(\lambda_1^2\beta_1^2+\lambda_2^2\beta_2^2)+o(\lambda^3), \tag{5.5}$$

in which $\lambda_i = a/R_i$ and $\beta_i = b_i/R_i$. This correctly reduces to the leading term of (4.4) for a circular duct, $R_1 = R_2 = R_o$, $b_1 = b$, $b_2 = 0$.

It is possible to demonstrate that the error term in (5.5) is of $O(\lambda^5)$. Indeed, it is easily possible to calculate this term explicitly by repeating the prior calculation of the 'quadratic' contribution $(\mathbf{v}'', p'')$ for the slightly altered boundary conditions on the sphere. We have not done this, however.

As the larger of the two radii becomes infinite, the additional pressure drop given by (5.5) goes to zero. This is, of course, to be expected since the additional energy dissipation remains finite, whereas the cross-sectional area becomes infinite. The additional pressure drop force, $\Delta P^+ A$, remains non-zero in this limit.

## 6. Ellipsoidal particles

The general methods of the present paper may be employed to calculate the additional pressure drop caused by the presence of a *non-spherical*, neutrally buoyant particle in a duct flow to terms of lowest order in $\lambda$—the ratio of particle to duct size; that is, the analogue of (5.1) can be developed for non-spherical particles. In this section the calculation will be carried to completion explicitly for a general ellipsoid with three unequal axes.

The term $\Delta P^+ V_m A$ gives the additional rate of mechanical energy dissipation due to the presence of the ellipsoid in the duct. As in the analogous case of the sphere, this quantity can be computed correctly to $O(\lambda^3)$ by retaining only the linear terms in (3.1); that is, by replacing the actual duct flow by an equivalent local shear flow characterized by $\mathbf{S}_o^o$. But Jeffery (1922, equation (59)) has already computed the additional dissipation rate $\dot{D}^+$ due to the presence of a neutrally buoyant ellipsoid in a general shearing flow. In this manner we obtain

$$\Delta P^+ V_m A = \dot{D}^+ + o(\lambda^3), \tag{6.1}$$

in which
$$\dot{D}^+ = \tfrac{16}{3}\pi\mu_o a_1 a_2 a_3 \left[\frac{\alpha_1'' S_{11}^2+\alpha_2'' S_{22}^2+\alpha_3'' S_{33}^2}{d} + 2S_{12}^2\frac{a_1 a_2}{\alpha_3'(a_1^2+a_2^2)} + 2S_{23}^2\frac{a_2 a_3}{\alpha_1'(a_2^2+a_3^2)} + 2S_{31}^2\frac{a_3 a_1}{\alpha_2'(a_3^2+a_1^2)}\right]. \tag{6.2}$$

Here, $a_1$, $a_2$, $a_3$ are the lengths of the semi-axes of the ellipsoid, and

$$\left.\begin{aligned} \alpha_1' &= \frac{(a_1 a_2 a_3)^2}{a_1}\int_0^\infty \frac{d\xi}{(a_2^2+\xi)(a_3^2+\xi)\Delta}, \quad \alpha_1'' = a_1 a_2 a_3 \int_0^\infty \frac{\xi\, d\xi}{(a_2^2+\xi)(a_3^2+\xi)\Delta}, \\ \Delta &= [(a_1^2+\xi)(a_2^2+\xi)(a_3^2+\xi)]^{\frac{1}{2}}, \quad d = \alpha_1''\alpha_2''+\alpha_2''\alpha_3''+\alpha_3''\alpha_1''. \end{aligned}\right\} \tag{6.3}$$

Expressions for the remaining $\alpha$ integrals can be obtained by permuting the indices.† The coefficients $S_{jk}$ refer to the components of $\mathbf{S}_o^o$ resolved along the principal axes of the ellipsoid, i.e. $\mathbf{S}_o^o = \mathbf{e}_j\mathbf{e}_k S_{jk}$, in which $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors along these axes. In the special case where the ellipsoid is one of revolution about the 1 axis (i.e. $a_2 = a_3$), with axis ratio $s = a_1/a_2$, the $\alpha$ integrals reduce to the forms

$$\alpha_1' = \frac{s^2}{4(s^2-1)^2}(3\theta+2s^2-5), \quad \alpha_2' = \alpha_3' = \frac{1}{(s^2-1)^2}(-3s^2\theta+s^2+2),$$

$$\alpha_1'' = \frac{s^2}{4(s^2-1)^2}[-(4s^2-1)\theta+2s^2+1], \quad \alpha_2'' = \alpha_3'' = \frac{s^2}{(s^2-1)^2}[(2s^2+1)\theta-3],$$

in which

$$\theta = \begin{cases} \dfrac{1}{s(s^2-1)^{\frac{1}{2}}}\cosh^{-1}s & (s>1), \\[2ex] \dfrac{1}{s(1-s^2)^{\frac{1}{2}}}\cos^{-1}s & (s<1). \end{cases} \tag{6.4}$$

To terms of dominant order the unidirectional flow in a duct may be regarded locally as a simple shear flow (Brenner 1966*a*, p. 391), the rate of shear $\kappa$ being given by the expression

$$\kappa = (2\mathbf{S}_o^o:\mathbf{S}_o^o)^{\frac{1}{2}} \equiv |(\nabla v^o)_o| \tag{6.5}$$

[cf. §5]. However, as is well known (Jeffery 1922; Goldsmith & Mason 1967), a neutrally buoyant ellipsoid of revolution immersed in a simple shearing flow undergoes a periodic rotation at the same time as its centre translates along a streamline at the same speed as the local fluid velocity $v_o^o$. For such a body, the $S_{jk}$ coefficients in (6.2) will therefore be periodic functions of time since $\mathbf{S}_o^o$ is constant with respect to axes fixed in space whereas the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ locked into the ellipsoid will vary periodically in time relative to a space-fixed observer. Accordingly, the pressure drop will be a periodic function of time. The time-average additional pressure drop $\langle\Delta P^+\rangle$ may be obtained from knowledge of the instantaneous value of $\dot{D}^+$ by integrating (6.1) over one period. The time-average additional dissipation rate $\langle\dot{D}^+\rangle$ has already been computed by Jeffery (1922), with the result that

$$\begin{aligned}\langle\Delta P^+\rangle V_m A &= \tfrac{4}{3}\pi\mu_o a_1 a_2^2 \kappa^2\left[\frac{2s^2}{(s^2-1)^2}\left\{\frac{s^2+1+2\hat{k}^2}{[(s^2+\hat{k}^2)(1+\hat{k}^2)]^{\frac{1}{2}}}-2\right\}\right.\\ &\quad\times\left\{\frac{\alpha_1''}{2\alpha_1'\alpha_2''}+\frac{1}{2\alpha_1'}-\frac{2s}{\alpha_2'(s^2+1)}\right\}+\frac{\hat{k}^2}{[(s^2+\hat{k}^2)(1+\hat{k}^2)]^{\frac{1}{2}}}\\ &\quad\left.\times\left\{\frac{1}{\alpha_1'}-\frac{2s}{\alpha_1'(s^2+1)}\right\}+\frac{2s}{\alpha_2'(s^2+1)}\right]+o(\lambda^3),\end{aligned} \tag{6.6}$$

in which $\hat{k} = k/a_2$, where $k$ is the 'orbit constant' defined by Jeffery. In effect, this constant specifies the initial orientation of the particle.

The above expression is valid for an ellipsoid of revolution in an arbitrary unidirectional duct flow, characterized by the shear rate (6.5). For a circular tube of radius $R_o$ we have $A = \pi R_o^2$, $\kappa = 4V_m b/R_o^2$, and $\lambda = \max(a_1, a_2)/R_o$.

† The dimensionless $\alpha$ integrals are related to the original $\alpha_o$, $\beta_o$, $\gamma_o$ integrals of Jeffery by the expressions

$$\|\alpha_1';\,\alpha_2';\,\alpha_3'\| = (a_1a_2a_3)^2\|\alpha_o'/a_1;\,\beta_o'/a_2;\,\gamma_o'/a_3\|,$$

$$\|\alpha_1'';\,\alpha_2'';\,\alpha_3''\| = a_1a_2a_3\|\alpha_o'';\,\beta_o'';\,\gamma_o''\|.$$

For a sufficiently small value of $\lambda$, the centre of the ellipsoid will maintain a fixed distance $b$ from the tube axis as the ellipsoid rotates and translates parallel to the tube axis. At larger values of $\lambda$, however, where wall effects are sensible, the centre of the ellipsoid is likely to migrate across the streamlines (Brenner 1966*a*, pp. 377 ff.), resulting in a time-dependent value of $\kappa$.

This research was supported by the National Science Foundation under grant numbers GK-1458 and GK-12583.

## REFERENCES

BRENNER, H. 1959 Dissipation of energy due to solid particles suspended in a viscous liquid. *Phys. Fluids*, **1**, 338–346.

BRENNER, H. 1962 Dynamics of a particle in a viscous fluid. *Chem. Engng Sci.* **17**, 435–446.

BRENNER, H. 1963 The Stokes resistance of an arbitrary particle. *Chem. Engng Sci.* **18**, 1–25.

BRENNER, H. 1964*a* The Stokes resistance of a slightly deformed sphere. *Chem. Engng Sci.* **19**, 519–539.

BRENNER, H. 1964*b* The Stokes resistance of an arbitrary particle: IV. Arbitrary fields of flow. *Chem. Engng Sci.* **19**, 703–727.

BRENNER, H. 1966*a* Hydrodynamic resistance of particles at small Reynolds numbers. In *Advances in Chemical Engineering*. Vol. 6 (T. B. Drew, J. W. Hoopes, Jr. and T. Vermuelen, eds.), pp. 287–438. Academic.

BRENNER, H. 1966*b* The Stokes resistance of an arbitrary particle: V. Symbolic operator representation of intrinsic resistance. *Chem. Engng Sci.* **21**, 97–109.

BRENNER, H. & HAPPEL, J. 1958 Slow viscous flow past a sphere in a cylindrical tube. *J. Fluid Mech.* **4**, 195–213.

BUNGAY, P. M. & BRENNER, H. 1970 Modelling of blood flow in the microcirculation. Tube flow of rigid particle suspensions. (To appear.)

CHEN, T. C. & SKALAK, R. 1970 Stokes flow in a cylindrical tube containing a line of spheroidal particles. *Appl. Sci. Res.* (in Press).

COX, R. G. & BRENNER, H. 1967 Effect of finite boundaries on the Stokes resistance of an arbitrary particle. III. Translation and rotation. *J. Fluid Mech.* **28**, 391–411.

EINSTEIN, A. 1905 Eine neue Bestimmung der Moleküldimensionen. *Ann. Phys.* (4) **19**, 289–306.

EINSTEIN, A. 1911 Berichtigung zu meiner Arbeit: Eine neue Bestimmung der Moleküldimensionen. *Ann. Phys.* (4) **34**, 591–592.

FAMULARO, J. & HAPPEL, J. 1965 Sedimentation of dilute suspensions in creeping motion. *Am. Inst. Chem. Engns J.* **11**, 981–988.

GOLDSMITH, H. L. & MASON, S. G. 1967 The microrheology of dispersions. In *Rheology: Theory and Applications, Vol.* 4 (F. R. Eirich, ed.), pp. 85–220. Academic.

GREENSTEIN, T. & HAPPEL, J. 1968 Theoretical study of the slow motion of a sphere and a fluid in a cylindrical tube. *J. Fluid Mech.* **34**, 705–710.

HABERMAN, W. L. & SAYRE, R. M. 1958 Motion of rigid and fluid spheres in stationary and moving liquids inside cylindrical tubes. *David Taylor Model Basin Rep.* no. 1143.

HIRSCHFELD, B. & BRENNER, H. 1971 Slow motion of a sphere in a cylindrical tube at an arbitrary angle of incidence. (To appear.)

HOCHMUTH, R. M. & SUTERA, S. P. 1970 Spherical caps in low Reynolds number tube flow. *Chem. Engng Sci.* **25**, 593–604.

JEFFERY, G. B. 1922 The motion of ellipsoidal particles immersed in a viscous fluid. *Proc. Roy. Soc.* A **102**, 161–179.

LAMB, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.

Ripps, D. L. & Brenner, H. 1967 The Stokes resistance of a slightly deformed sphere: II. Intrinsic resistance operators for an arbitrary initial flow. *Chem. Engng Sci.* **22**, 375–387.

Schowalter, W. R., Chaffey, C. E. & Brenner, H. 1968 Rheological behaviour of a dilute emulsion. *J. Colloid Interface Sci.* **26**, 152–160.

Sonshine, R. M. & Brenner, H. 1966 The Stokes translation of two or more particles along the axis of an infinitely long circular cylinder. *Appl. Sci. Res.* **16**. 425–454.

Wang, H. & Skalak, R. 1969 Viscous flow in a cylindrical tube containing a line of spherical particles. *J. Fluid Mech.* **38**, 75–96.